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At the Other Side of a Saddle-Node

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Abstract. We describe phenomena occurring just before a saddle-node bifurcation for one-parameter families of interval maps. In particular, as a parameter approaches the bifurcation value, attracting periodic orbits of periods $k, k+1, k+2, k+3, \ldots$ can appear. We make a detailed study of a family of "cusp-shaped" maps, where this phenomenon occurs in a pure form.

1. Introduction

For a one parameter family f_μ of maps of an interval into itself, a saddle-node bifurcation occurs when the graph of f_μ (or f_μ^n) touches the diagonal and then crosses it. A fixed point and immediately after it – a pair of fixed points (respectively a periodic point of period n and then a pair of them) appears; one of these points is attracting and the other one repelling. However, here we will not be interested in these fixed (periodic) points. Instead, we shall look what happens at the other side of the saddle-node bifurcation, i.e. for these parameters for which a fixed point is not created yet.

This situation has been considered by Newhouse, Palis, and Takens in [NPT]. However, their aim was different than ours and hence [NPT] does not contain explicit statements of the results interesting to us. We shall restate (and reprove) these results in the form showing clearly what is going on.

The main phenomenon that may be observed in some families, is period adding. As the parameter approaches the bifurcation value, attracting periodic orbits of period k, $k+1$, $k+2$, $k+3, \ldots$ (or k, $k+n$, $k+2n$, $k+3n, \ldots$ if the bifurcation occurs for f_μ^n) appear. However, in many cases there are many other periods of attracting periodic orbits which appear in the considered interval of parameters. This has to happen for instance if the maps f_μ are smooth and unimodal (see [MSS, CE]). This is why we turned to the investigation of unimodal maps which are smooth except at the critical point, where the derivative is discontinuous (and bounded away from zero).

Such "cusp-shaped" maps appear in many experimental or model systems, e.g. the Lorenz model [L], models of flip-flop process in visual perception [AB] or a