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Zonal Schrödinger Operators on the n-Sphere: Inverse Spectral Problem and Rigidity

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Abstract. We study the Direct and Inverse Spectral Problems for a class of Schrödinger operators $H = -\Delta + V$ on S_n with *zonal* (axisymmetric) potentials. Spectrum of H is known to consist of clusters of eigenvalues $\{\lambda_{km} = k(k+n-1) + \mu_{km}: m \leqq k\}$. The main result of the work is to derive asymptotic expansion of spectral shifts $\{\mu_{km}\}$ in powers of k^{-1}, and to link coefficients of the expansion to certain transforms of V. As a corollary we solve the Inverse Problem, get explicit formulae for the Weinstein *band-invariants* of cluster distribution measures, and establish *local spectral rigidity* for zonal potential. The latter provides a partial answer to a long standing Spectral Rigidity Hypothesis of V. Guillemin.

The Direct/Inverse Spectral Problems for differential operators ask for connections between the "geometric/dynamical" data (like coefficients of the operator, Riemannian metric, potential, or geometry of the region) on one hand, and its "spectral data" (eigenvalues/eigenfunctions) on the other. The well known examples include

i) M. Kac: "shape of the drum" problem determination (modulo rigid motions) of the boundary of a region from eigenvalues of the Laplacian.
ii) The "shape of the metric" problem ([MS]), which poses a similar question for the Riemannian metric of manifold and the Laplace–Beltrami operator.
iii) Potential (perturbation) problem for Schrödinger operators $H = -\Delta + V(x)$. Here the manifold and metric (hence the Laplacian) are fixed, and one studies the connection between spec(H) and the potential function V.

Our work belongs to the third type.

The classical and best studied example of perturbation problems are the regular Sturm–Liouville (S–L) operators $H = -\partial^2 + V(x)$ on $[0;1]$, with various types of boundary conditions: 2-point; periodic Floquet, etc. Such operators have simple (multiplicity free) spectrum $\{\lambda_k\}$, and their asymptotics were known since the old