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Toda Lattice Hierarchy and Conservation Laws

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Abstract. Time evolutions of the Toda lattice hierarchies of Ueno and Takasaki are induced by Hamiltonians which are conservation laws for the original (one and two dimensional) Toda lattice obtained by Olive and Turok. Moreover these Hamiltonians for two dimensional Toda lattice hierarchy are also conserved quantities of the two component KP hierarchy in which that system is embedded. The one dimensional Toda lattice hierarchy is characterized by the bilinear relations, and a new version of the one dimensional Toda lattice hierarchy is constructed. Generalized Toda lattice hierarchies associated to all affine Lie algebras are presented.

0. Introduction

The Toda lattice has been, together with the Kortweg-de Vries (KdV) equation, one of the most important completely integrable non-linear systems, many features of which have been revealed by various methods. In this paper we will investigate the interrelationship between the Toda lattice hierarchy (K. Ueno, K. Takasaki [U–T]) and the structure of the Toda lattice as a Hamiltonian system with infinitely many constants of the motion (D. Olive, N. Turok [O–T1, 2, 3]), and conserved quantities of the multi component KP hierarchy. We also give a characterization by the bilinear relations and another version of one dimensional Toda lattice hierarchy, and present a candidate for the generalized Toda lattice hierarchy associated to any affine Lie algebra.

Ueno and Takasaki [U–T] introduced the Toda lattice hierarchy (hereafter we will abbreviate it to TL hierarchy), inspired by the theory of the Kadomtsev–Petviashvili (KP) hierarchies, and investigated its Lax representation, Zakharov–Shabat representation, the linearization, the τ function and its bilinear equations of Hirota-type etc., and showed that the TL hierarchy is embedded into the 2 component KP hierarchy. They also defined the periodic reduction and the restriction of the system to the one dimensional sector.

On the other hand, Olive and Turok [O–T1, 2, 3] made full use of the classical r matrix (they call it the **P** operator; cf. [Fa, Fa–T]) introduced from the theory