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Navier–Stokes Equations and Area of Interfaces

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Abstract. We present new a priori estimates for the vorticity of solutions of the three dimensional Navier–Stokes equations. These estimates imply that the L^1 norm of the vorticity is a priori bounded in time and that the time average of the $4/(3+\varepsilon)$ power of the $L^{4/(3+\varepsilon)}$ spatial norm of the gradient of the vorticity is a priori bounded. Using these bounds we construct global Leray weak solutions of the Navier–Stokes equations which satisfy these inequalities. In particular it follows that vortex sheet, vortex line and even more general vortex structures with arbitrarily large vortex strengths are initial data which give rise to global weak solutions of this type of the Navier–Stokes equations. Next we apply these inequalities in conjunction with geometric measure theoretical arguments to study the two dimensional Hausdorff measure of level sets of the vorticity magnitude. We obtain a priori bounds on an average such measure, $\langle \mu \rangle$. When expressed in terms of the Reynolds number and the Kolmogorov dissipation length η, these bounds are

$$\langle \mu \rangle \leqq \frac{L^3}{\eta}(1 + \mathrm{Re}^{-1/2})^{1/2}.$$

The right-hand side of this inequality has a simple geometrical interpretation: it represents the area of a union of non-overlapping spheres of radii η which fill a fraction of the spatial domain. As the Reynolds number increases, this fraction decreases. We study also the area of level sets of scalars and in particular isotherms in Rayleigh–Benard convection. We define a quantity, $\langle \mu \rangle_{r,t}(x_0)$ describing an average value of the area of a portion of a level set contained in a small ball of radius r about the point x_0. We obtain the inequality

$$\langle \mu \rangle_{r,t}(x_0) \leqq C\kappa^{-1/2} r^{5/2} \langle v(x_0) \rangle^{1/2},$$

where κ is the diffusivity constant and $\langle v \rangle$ is the velocity. The inequality is valid for r small but larger than a small scale $\lambda = \kappa \langle v \rangle^{-1}$. In the case of turbulent velocities this scale is smaller than the smallest significant physical scale,

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