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Measure and Dimension of Solenoidal Attractors of One Dimensional Dynamical Systems

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Abstract. Let $f: M \to M$ be a C^∞-map of the interval or the circle with non-flat critical points. A closed invariant subset $A \subset M$ is called a solenoidal attractor of f if it has the following structure: $A = \bigcap_{n=1}^{\infty} \bigcup_{k=0}^{p_n - 1} I_k^{(n)}$, where $\{I_k^{(n)}\}_{k=0}^{p_n}$ is the cycle of intervals of period $p_n \to \infty$. We prove that the Lebesgue measure of A is equal to zero and if $\sup(p_{n+1}/p_n) < \infty$ then the Hausdorff dimension of A is strictly less than 1.

1. Introduction

Let M be a one dimensional compact manifold with boundary, i.e. a finite union of disjoint intervals and circles. Let us consider the class $\mathfrak{A}$ of C^∞-smooth transformations $f: M \to M$ with non-flat critical points [the last means that for each critical point c there exists n such that $f^{(n)}(c) \neq 0$]. The map f is called d-modal if it has d extrema in $\operatorname{int} M$ (for $d=1$ f is said to be unimodal). Let $f^n = f \circ f \circ \ldots \circ f$ denote the n^{th} iterate of f.

By *solenoid attractor* of M (or simply a *solenoid*) we mean a closed f-invariant subset $A \subset M$ of the following structure:

$$A = \bigcap_{n=1}^{\infty} M^{(n)}, \qquad M^{(1)} \supset M^{(2)} \supset \ldots, \tag{1}$$

where

$$M^{(n)} = \bigcup_{k=0}^{p_n - 1} I_k^{(n)} \tag{2}$$

is the union of p_n closed disjoint intervals $I_k^{(n)}$ such that $f I_k^{(n)} \subset I_{k+1}^{(n)}$ (here $I_{p_n}^{(n)}$ is identified with $I_0^{(n)}$), $p_n \to \infty$.

Clearly, p_n is a divisor of p_{n+1}. The *type* of the solenoid A is the maximal possible sequence $\{p_n\}_{n=1}^{\infty}$ of the pairwise distinct periods p_n.

Let λ denote the Lebesgue measure on M and $\dim X$ denote the Hausdorff dimension of a subset $X \subset M$. The aim of the present paper is to prove the following theorem: