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Supersymmetry and the Möbius Inversion Function

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Abstract. We show that the Möbius inversion function of number theory can be interpreted as the operator $(-1)^F$ in quantum field theory. Consequently, we are able to provide physical interpretations for various properties of the Möbius inversion function. These include a physical understanding of the Möbius Inversion Formula and of a result that is equivalent to the prime number theorem. Supersymmetry and the Witten index play a central rôle in these constructions.

1. Introduction

One of the most fundamental functions of number theory is the Möbius inversion function μ [3]. It is a function whose domain is the positive integers, and which is defined as follows. Let us say that an integer is *squarefree* if it is divisible by no perfect square other than 1, which of course means that in the prime decomposition of a squarefree number, each prime factor present appears exactly once. Then the function μ is defined by

$$\mu(n) = \begin{cases} +1 & \text{if } n \text{ is squarefree with an even number of prime factors} \\ -1 & \text{if } n \text{ is squarefree with an odd number of prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

The function μ appears throughout number theory. For example, it plays a central rôle in the theory of Dirichlet convolution, and arises as well in several proofs of the prime number theorem. We will see in this paper that the function μ has a very natural physical interpretation. In the proper context, it is equivalent to $(-1)^F$, the operator that distinguishes fermionic from bosonic states and operators, with the fact that $\mu(n) = 0$ when n is not squarefree being equivalent to the Pauli exclusion principle.

In this paper, we develop this identification between μ and $(-1)^F$. In so doing, we will be able to use physical arguments to derive and understand some of the fundamental results of arithmetic number theory. We will see in particular that