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Scaling for the Discrete Mathieu Equation

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Abstract. The scaling of the total width of the band for the discrete Mathieu equation is studied in the critical region near the transition between localized and extended states, for the special case in which there is one period of the modulation for p lattice spacings. A general expression for the bandwidth W in the critical region is found. At the critical point an analytic expression for pW is found which agrees to one part in 10^8 with the result deduced from numerical work.

1. Introduction

In an earlier paper [1], which I refer to as BW, a study was made of the total band width for the discrete Mathieu equation, which can be written as

$$V_1 a_{n-1} + 2V_2 \cos(2\pi n\phi + k_2) a_n + V_1 a_{n+1} = E a_n; \tag{1}$$

to simplify the subsequent discussion I take V_1 and V_2 to be positive. From the work of Aubry and André [2] it was already known that this equation has a critical point for $V_1 = V_2$, where, in the incommensurate limit with ϕ irrational, all eigenfunctions become extended instead of localized, and the total bandwidth, again in the incommensurate limit, appears to be equal to $4|V_2 - V_1|$, so that the bandwidth vanishes at the critical point. In BW I gave a derivation of this linear behaviour of the bandwidth, and did some numerical studies of the way the bandwidth behaves when the common period, p, of the sinusoidal term and the lattice becomes large. In this commensurate case, with

$$\phi = q/p, \tag{2}$$

the bandwidth is defined as the measure of the union over all values of the phase k_2 of the spectrum of Eq. 1, which consists of a set of p bands which are generally distinct, except at $E = 0$, where there are two bands touching for even values of p. I found what appeared to be a scaling behavior of the bandwidth in the critical

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