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The Weil-Petersson Geometry of the Moduli Space of $SU(n \geqq 3)$ (Calabi-Yau) Manifolds I

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Abstract. The Weil-Petersson metric is defined on the moduli space of Calabi-Yau manifolds. The curvature of this Weil-Petersson metric is computed and its potential is explicitely defined. It is proved that the moduli space of Calabi-Yau manifolds is unobstructed (see Tian).

Dedicated to Lipman Bers on the occasion of his 75th birthday

0.1. Introduction

In this paper we are going to study some differential-geometric properties of the moduli space of compact complex manifolds of $\dim_{\mathbb{C}} \geqq 3$ which admit non-flat metrics g with holonomy groups $H(g) \neq \{0\}$ and $H(g) \subseteqq SU(n)$. Such manifolds we will call $SU(n)$ or Calabi-Yau manifolds. Before stating the main results, we will make several remarks.

Remark 0.1.1. It is not difficult to see that a metric on a compact complex manifold whose holonomy $H^0 \neq \{0\}$ and $H \subseteqq SU(n)$, will be Kähler and Ricci flat. We will call it the Calabi-Yau metric. (See [2]).

Remark 0.1.2. If M is a Calabi-Yau manifold, then from the theory of invariants of the group $SU(n)$ and the fact that the holonomy group $H^0 \neq \{0\}$ and $H \subseteqq SU(n)$, it follows $H^0(M, \Omega^i) = 0$ for $1 < i < n$ and $H^0(M, \Omega^n)$ is spanned by a holomorphic n-form w_0, which has no zeroes and no poles. This implies that $c_1(M) = 0$. Constructions of Calabi-Yau manifolds are based on the solution of the Calabi conjecture by Yau. See [15].

Recently $SU(3)$ manifolds have attracted the interest of physicists working on string theory and algebraic geometers working on the classification of threefolds and on algebraic cycles.

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