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Metastable States for the Becker–Döring Cluster Equations

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Abstract. The Becker–Döring equations, in which $c_l(t)$ can represent the concentration of l-particle clusters or droplets in (say) a condensing vapour at time t, are

$$dc_l(t)/dt = J_{l-1}(t) - J_l(t) \quad (l = 2, 3, \ldots)$$

with

$$J_l(t) := a_l c_1(t) c_l(t) - b_{l+1} c_{l+1}(t)$$

and either $c_1 = \text{const.}$ ('case A') or $\rho := \sum_1^\infty l c_l = \text{const.}$ ('case B'). The equilibrium solutions are $c_l = Q_l z^l$, where $Q_l := \prod_2^l (a_{r-1}/b_r)$. The density of the saturated vapour, defined as $\rho_s := \sum_1^\infty l Q_l z_s^l$, where z_s is the radius of convergence of the series, is assumed finite. It is proved here that, subject to some further plausible conditions on the kinetic coefficients a_l and b_l, there is a class of "metastable" solutions of the equations, with $c_1 - z_s$ small and positive, which take an exponentially long time to decay to their asymptotic steady states. (An "exponentially long time" means one that increases more rapidly than any negative power of the given value of $c_1 - z_s$ (or, in case B, $\rho - \rho_s$) as the latter tends to zero). The main ingredients in the proof are (i) a time-independent upper bound on the solution of the kinetic equations (this upper bound is a steady-state solution of case A of the equations, of the type used in the Becker–Döring theory of nucleation), and (ii) an upper bound on the total concentration of particles in clusters greater than a certain critical size, which (with suitable initial conditions) remains exponentially small until the time becomes exponentially large.

1. Introduction

In 1979 an article entitled "towards a rigorous theory of metastability" was published by J. L. Lebowitz and this author (Penrose and Lebowitz 1979, 1987).