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Optimal Bounds for Ratios of Eigenvalues of One-Dimensional Schrödinger Operators with Dirichlet Boundary Conditions and Positive Potentials

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Abstract. Consider the Schrödinger equation $-u'' + V(x)u = \lambda u$ on the interval $I \subset \mathbb{R}$, where $V(x) \geqq 0$ for $x \in I$ and where Dirichlet boundary conditions are imposed at the endpoints of I. We prove the optimal bound

$$\frac{\lambda_n}{\lambda_1} \leqq n^2 \quad \text{for} \quad n = 2, 3, 4, \ldots$$

on the ratio of the n^{th} eigenvalue to the first eigenvalue for this problem. This leads to a complete treatment of bounds on ratios of eigenvalues for such problems. Extensions of these results to singular problems are also presented. A modified Prüfer transformation and comparison techniques are the key elements of the proof.

1. Introduction

The Schrödinger operator $H = -\Delta + V(x)$ acting on $L^2(\Omega)$ with Dirichlet boundary conditions is known to have purely discrete spectrum if Ω is a bounded connected subset of $\mathbb{R}^d$ with smooth boundary. We denote the eigenvalues listed in ascending order (with multiplicities included) by $\{\lambda_i\}_{i=1}^{\infty}$. Furthermore, $\lambda_2 > \lambda_1$ (nondegeneracy of the groundstate) and, if $V(x) \geqq 0$ for all $x \in \Omega$, $\lambda_1 > 0$. Thus

$$0 < \lambda_1 < \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \ldots, \tag{1.1}$$

and in this context it makes sense to consider the boundedness of the ratio λ_2/λ_1 or, more generally, of λ_n/λ_1. Such questions seem to have first been addressed by Payne et al. [16, 17] who considered the case where $V \equiv 0$ and $\Omega \subset \mathbb{R}^2$ (the stretched membrane problem). Among other things they proved the bounds

$$\frac{\lambda_2}{\lambda_1} \leqq 3 \tag{1.2}$$

and, more generally,

$$\frac{\lambda_{n+1}}{\lambda_n} \leqq 3. \tag{1.3}$$