Commun. Math. Phys. 122, 531–562 (1989)

Communications in Mathematical Physics

Deformation Quantization of Heisenberg Manifolds

Marc A. Rieffel*

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract. For M a smooth manifold equipped with a Poisson bracket, we formulate a C^*-algebra framework for deformation quantization, including the possibility of invariance under a Lie group of diffeomorphisms preserving the Poisson bracket. We then show that the much-studied non-commutative tori give examples of such deformation quantizations, invariant under the usual action of ordinary tori. Going beyond this, the main results of the paper provide a construction of invariant deformation quantizations for those Poisson brackets on Heisenberg manifolds which are invariant under the action of the Heisenberg Lie group, and for various generalizations suggested by this class of examples. Interesting examples are obtained of simple C^*-algebras on which the Heisenberg group acts ergodically.

About a decade ago a new approach to the quantization of classical mechanical systems was introduced by Vey [30], and Flato, Fronsdal, Lichnerowicz and coauthors [3, 12]. Their approach involves viewing quantization as a deformation of structure, and goes roughly as follows. A classical mechanical system is given by its phase space, which is a C^∞-manifold M, together with a symplectic structure which defines a Poisson bracket $\{\ ,\ \}$. To quantize this system, one selects a suitable algebra A of C^∞ functions on M (functions of compact support, Schwartz functions, polynomials, to the extent these make sense), with the product being pointwise multiplication. One then deforms this product "in the direction of" the Poisson bracket. That is, if we denote the deformation parameter by "Planck's constant" $\hbar$, taking real values in some interval about 0, then one tries to define a family $*_\hbar$ of associative but not necessarily commutative products parametrized by $\hbar$, in such a way that for $f, g \in A$ one has

$$f *_\hbar g \to fg$$

* This work was supported in part by National Science Foundation grant DMS 8601900