Commun. Math. Phys. 122, 267–280 (1989)

Communications in Mathematical Physics

Nahm's Transformation for Instantons

Peter J. Braam[1,*] and Pierre van Baal[2]

[1] Merton College, Oxford OX1 4JD, United Kingdom

[2] CERN, Theory Division, CH-1211 Geneva 23, Switzerland

Abstract. We describe in mathematical detail the Nahm transformation which maps anti-self dual connections on the four-torus $(S^1)^4$ onto anti-self-dual connections on the dual torus. This transformation induces a map between the relevant instanton moduli spaces and we show that this map is a (hyperKähler) isometry.

Introduction

This paper deals with "magical" properties of $U(n)$ anti-self-dual (asd) connections A on a C^n—bundle F over a four-torus T^4. The "witchcraft" starts by introducing a family of Dirac operators coupled to (F, A) parametrized by the dual torus $\hat{T}^4$. The families index turns out to be a bundle $\hat{F} \to \hat{T}^4$ (under a genericity assumption on A) and comes equipped with a natural connection $\hat{A}$, which is again asd (Theorem 1.5). This is Nahm's transform. Doing it again to $(\hat{F}, \hat{A})$, we obtain $(\hat{\hat{F}}, \hat{\hat{A}})$ and a unitary equivalence $(F, A) \sim (\hat{\hat{F}}, \hat{\hat{A}})$. In other words the square of Nahm's transform is the identity (Theorem 2.8 & 2.9). This was discovered by the authors and independently by Schenk [21], and relies heavily on some ideas of Nahm [19, 20]. The transformation now induces a map of moduli spaces of (generic) asd connections $N: \mathcal{M}(F) \to \mathcal{M}(\hat{F})$. The spaces $\mathcal{M}(F)$ and $\mathcal{M}(\hat{F})$ carry a hyperKähler metric and N turns out to be a hyperKähler isometry, as was conjectured by S. K. Donaldson (Theorem 3.4).

This transformation has been around for awhile, and the fact that its square is the identity was announced by Nahm [19, 20] in the early eighties (see also Corrigan and Goddard [6]). However, for mathematicians it is not so easy to understand Nahm's work. In a way, the torus case treated here is the simplest version of Nahm's transform. Nahm originally developed his transformation for instantons invariant under subgroups of R^4, different from the 4-dimensional lattice (such as R or Z). For time-invariant instantons this was used extensively by Hitchin

* Address until September 1, 1989, Department of Mathematics University of Utah, Salt Lake City, Utah 84112, USA