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Inequalities and Many Phase Transitions in Ferromagnetic Systems

R. Fernández[1] and J. Slawny[2]

[1] Department of Mathematics, University of Texas at Austin, Austin, TX 78731, USA
[2] Center for Transport Theory and Mathematical Physics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

Abstract. A general construction of ferromagnetic systems with many phase transitions is given. It is based on two new results: an extension of one of the GKS inequalities to not necessarily ferromagnetic interactions, and a uniqueness of the Gibbs state theorem for perturbations of some simple systems *at all temperatures*.

1. Introduction

This work arose from an attempt to understand the global structure of phase diagrams of classical ferromagnetic lattice systems. Several techniques have been developed to study the *low temperature* phase diagrams; among them the Peierls argument [27, 19, 6], reflection-positivity techniques [14–16], Pirogov-Sinai theory [29, 34, 32] and its extensions [3, 9, 25, 34, 4, 5], and algebraic methods peculiar to ferromagnetic systems [21, 23, 28, 32, 26]. However, very little is known about how this phase diagram changes with the temperature, especially in systems with multispin interactions. Here we explore a general mechanism explaining occurrence of many phase transitions in some ferromagnetic systems as the temperature is varied. In particular we construct models for which the existence of several phase transitions can be proved. Later, in this introduction and in Sect. 5.4, we comment on the relation of this work to an earlier related work by Pfister [28].

The systems considered here are ferromagnetic in the sense that the configuration space is an (abelian) group and the Hamiltonians H are negative definite, i.e. they are linear combinations of characters with negative coefficients. Such systems have a distinguished Gibbs state $\langle\ \rangle^+$ which breaks the symmetry of the Hamiltonian in a maximal way. Let $\mathscr{A}_\beta(H)$ denote the set of characters σ for which $\langle\sigma\rangle^+_{\beta H} \neq 0$, where β is the inverse temperature. By a general argument, [30, 28], $\mathscr{A}_\beta$ is becoming larger as β increases, it always contains the group $\bar{\mathscr{B}}(H)$ generated by the characters appearing in H and it stabilizes both at low and high temperatures, to $\mathscr{A}_\infty(H)$ and to $\mathscr{A}_0(H)$ $(=\bar{\mathscr{B}}(H))$ respectively. If $\mathscr{A}_\infty(H)$ is strictly larger than $\mathscr{A}_0(H)$ one has a phase transition as the temperature is varied. The problem is to investigate at how many temperatures $\mathscr{A}_\beta(H)$ is changing.