Commun. Math. Phys. 120, 613–641 (1989)

Communications in Mathematical Physics

The Classification of Monopoles for the Classical Groups

Jacques Hurtubise*

Institute for Advanced Study Princeton, NJ 08540, USA and Department of Mathematics, McGill University, Burnside Hall, 805 Sherbrooke St. W. Montréal, Québec, Canada H3A 2K6

Abstract. By studying a construction of Nahm, we compute the moduli spaces of monopoles with maximal symmetry breaking at infinity for $SU(N)$, $SO(N)$ and $Sp(N)$; these are found to be equivalent to spaces of holomorphic maps from $\mathbb{P}_1$ into flag manifolds.

Introduction

Let P be a principal G-bundle over $\mathbb{R}^3$, G a compact group, ∇ a connection on P with curvature F, φ (the "Higgs field") a section of $\mathrm{ad}(P)$, the associated adjoint bundle: (∇, φ) is a *monopole* if it solves the Bogomoln'yi equation, $F = *\nabla\varphi$, and if it satisfies the boundary condition of having finite action, with φ tending toward a finite limit at infinity, with values in a fixed G-orbit in $\mathrm{ad}(P)$. Such monopoles, particularly for the group $SU(2)$, have been extensively studied in recent years, from various points of view [JT, Hi, Mu]. One particularly successful construction, due to Nahm [N], describes these monopoles in terms of solutions to some non-linear ordinary differential equations, Nahm's equations. A theorem, whose full proof is due to Hitchin [Hi], shows that for $SU(2)$, there is a natural equivalence between $SU(2)$ monopoles and an appropriate class of solutions to Nahm's equations. Using this, Donaldson was able to give a description of the moduli space of $SU(2)$ monopoles:

Theorem [D1]. *Given an isomorphism $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$, compatible with the usual metrics there is a natural correspondence between a circle bundle $\tilde{M}_k$ defined over the moduli space of $SU(2)$ monopoles of charge k, and the complex manifold R_k of rational maps $f: \mathbb{P}_1 \to \mathbb{P}_1$ of degree k, with $f(\infty) = 0$.*

In terms of the monopole, the extra circle corresponds to the choice of a framing at infinity; see [AHi, Hu].

Recently, in [HuM], a proof was given of the validity of Nahm's construction for all the classical groups, for monopoles with maximal symmetry breaking at infinity. This condition means that if G is the gauge group with maximal torus T,

---

* Research supported in part by NSERC grant A8361 and FCAR grant EQ3518