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Fine Tuning of Resonances and Periodic Solutions of Hamiltonian Systems Near Equilibrium

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1. Introduction

Consider in R^{2n} a Hamiltonian System which has the origin as isolated equilibrium. We use the notation $z_k \equiv (q_k, p_k)$, $k = i \cdots n$, and we assume that the Hamiltonian H is of class C^2 and that H_2, the quadratic part of H, is given by

$$H_2 = 1/2 \sum_0^n v_i z_i^2, \quad z_i^2 = q_i^2 + p_i^2, \tag{1.1}$$

with $v_i \neq 0$, $i = 1 \cdots n$. If $v_i \cdot v_j^{-1}$ is not an integer for any pair i,j $(i \neq j)$ a theorem by Lyapunov [1] states that one can find a neighborhood N of the origin such that in N there are n families of elliptic periodic solutions of (1.1); the kth family has minimal periods approximately equal to $2\pi v_k^{-1}$ and lies approximately on the hyperplane $\{z | z_h = 0 \text{ if } h \neq k\}$. One can use as parameter the distance from the origin. These results follow from an application of the Inverse Function Theorem to the periodicity condition.

General lower bounds on the number of periodic solutions on every energy surface sufficiently close to the origin have been obtained within a variational approach. These results hold independently of whether there is a resonance among the frequencies, but are often restricted to the case when H_2 is of definite sign [2] (see however [6]); also, the localization and stability of the periodic solutions are in general not known.

On the other hand, if $v_i = k v_j$ for some $i \neq j$, $k \in \mathbf{Z}$ (a resonant case) the number of families can be different from the one indicated in Lyapunov's Theorem, and some of the periodic solutions are hyperbolic. Systems with resonance have been studied extensively in phase space [3]. [4] leading to various estimates on the number of families of periodic solutions and their stability. In particular it was shown (see also [5]) that, if $n = 2$, $v_2 = 2v_1$, one has in general three families; the minimal period is approximately $2\pi v_1^{-1}$ for two of them, while it is approximately

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