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Lie Group Exponents and $SU(2)$ Current Algebras

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Abstract. Due to the Cappelli-Itzykson-Zuber classification, the minimal conformally invariant quantum field theories with $SU(2)$ currents are classified by the ADE Lie algebras. Here I give a conceptual proof of the empirically valid relation between their partition functions and the Lie algebra exponents.

Consider a conformally invariant quantum field theory on $S_1 \times R$ with left and right $SU(2)$ currents. Let the Hilbert space of the theory decompose into a finite number of irreducible level k representations of the Kac-Moody current algebra $A_1^{(1)} \times A_1^{(1)}$. Then the partition function is of the form

$$Z(w, \bar{w}) = \sum_{i,j=1}^{k+1} \chi_i^{(k)}(w) a_{ij} \chi_j^{(k)}(w)^*, \tag{1}$$

where the a_{ij} are non-negative integers. The $\chi_i^{(k)}$, $i = 1, \ldots, k+1$ are the characters of the irreducible unitary positive energy representations of level k of $A_1^{(1)}$. The label i is the dimension of the subspace of lowest energy, which forms an irreducible representation of the $SU(2)$ charge algebra.

If the vacuum state is non-degenerate, one must have $a_{11} = 1$. As the partition function can be written as a functional integral over a torus, it must be invariant under modular transformations. The partition functions of this type are in one-to-one correspondence with the Lie algebras of ADE type Lie groups G. In particular, $k+2$ is the Coxeter number of G, and a_{ii} is the number of G exponents equal to i [1].

So far this fact had not been explained in a conceptual way, though in [2] I gave the following construction of the $SU(2)$ modular invariants in terms of G. For a given G fix a set $\Delta^+(G)$ of positive roots and consider the subgroup H of G which leaves the highest root α invariant. Moreover, consider the $SU(2)$ subgroup of G generated by E_α and $E_{-\alpha}$. The coset space $G/(H \times SU(2))$ is the unique quaternionic symmetric space with symmetry group G. More precisely, for adjoint groups G, H the holonomy group is $(\tilde{H} \times SU(2)/Z_2$, where $\tilde{H}$ is a double cover of H. Ignoring