Commun. Math. Phys. 118, 99–118 (1988)

Communications in
Mathematical
Physics

A Geometrical Approach of Quasiperiodic Tilings

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Abstract. Tilings provide generalized frames of coordinates and as such they are used in different areas of physics. The aim of the present paper is to present a unified and systematic description of a class of tilings which have appeared in contexts as disconnected as crystallography and dynamical systems. The tilings of this class show periodic or quasiperiodic ordering and the tiles are related to the unit cube through affine transformations. We present a section procedure generating canonical quasiperiodic tilings and we prove that true tilings are indeed obtained. Moreover, the procedure provides a direct and simple characterization of quasiperiodicity which is suitable for tilings but which does not refer to Fourier transform.

I. Introduction

The present paper is devoted to the construction and analysis of two kinds of tilings of the euclidean space $\mathbf{R}^n$: 1. "Oblique" periodic tilings. 2. "Canonical" quasiperiodic tilings, as we may call them.

The tilings of the first kind, the *oblique* tilings, are periodic in the sense that they are invariant under the action of the discrete subgroup $\mathbf{Z}^n$ of $\mathbf{R}^n$. The oblique tiles depend on two orthogonal complementary subspaces E, of dimension d, and E' of $\mathbf{R}^n$ which are given a priori with an arbitrary, albeit fixed, orientation with respect to the lattice. Every oblique tile a "rectangle," that is the direct sum of a polyhedron in E and a polyhedron in E'; the polyhedron in E is the projection (onto E) of a d-dimensional facet (a d-facet, for short) of the n-dimensional unit cube; the polyhedron in E' is, similarly but up to a sign, the projection (onto E') of the $(n-d)$-facet which is complementary to the previous one. If one proceeds in this way for all the $\binom{n}{d}$ differently oriented facets having the same dimension as E, one builds a set of $\binom{n}{d}$ oblique polytopes in $\mathbf{R}^n$ which form a partition of a fundamental cell for $\mathbf{Z}^n$ and thereby provide a set of so-called "prototiles" for the tiling. This is stated and proved in our main theorem in Sect. III.