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Global Existence of Smooth Solutions and Stability of Solitary Waves for a Generalized Boussinesq Equation

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Abstract. Certain generalizations of one of the classical Boussinesq-type equations,

$$u_{tt} = u_{xx} - (u^2 + u_{xx})_{xx}, \tag{$*$}$$

are considered. It is shown that the initial-value problem for this type of equation is always locally well posed. It is also determined that the special, solitary-wave solutions of these equations are nonlinearly stable for a range of their phase speeds. These two facts lead to the conclusion that initial data lying relatively close to a stable solitary wave evolves into a global solution of these equations. This contrasts with the results of blow-up obtained recently by Kalantarov and Ladyzhenskaya for the same type of equation, and casts additional light upon the results for the original version (∗) of this class of equations obtained via inverse-scattering theory by Deift, Tomei and Trubowitz.

1. Introduction

In the 1870's, Boussinesq derived some model equations for the propagation of small amplitude, long waves on the surface of water. These equations possess special, travelling-wave solutions called solitary waves, and Boussinesq's theory was the first to give a satisfactory, scientific explanation of the phenomenon of solitary waves discovered by Scott-Russell and reported more than thirty years earlier. In one of his papers (Boussinesq, 1872) he also proposed what we would now call a Lyapunov function, which he argued was connected to the stability of these solitary waves.

The original equation due to Boussinesq is not the only mathematical model for small-amplitude, planar, long waves on the surface of water. Different choices of the dependent variables, plus the possibility of modifying lower order terms by the use of the leading order relationships can lead to a whole range of equations,

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