Commun. Math. Phys. 112, 633–638 (1987)

Communications in Mathematical Physics

Conformal Invariants for Determinants of Laplacians on Riemann Surfaces

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Abstract. For a Riemann surface with smooth boundaries, conformal (Weyl) invariant quantities proportional to the determinant of the scalar Laplacian operator are constructed both for Dirichlet and Neumann boundary conditions. The determinants are defined by zeta function regularization. The other quantities in the invariants are determined from metric properties of the surface. As applications explicit representations for the determinants on the flat disk and the flat annulus are derived.

I. Introduction

It is ancient lore that one can find the inverses of Laplace operators (Green's functions) on Riemann surfaces by conformal mappings from geometries where the potential theory problem is soluble directly. In this article I show that an analogous result applies to the calculation of determinants of Laplacians. By studying the variation under Weyl transformations of determinants of the scalar Laplacian on Riemann surfaces with smooth boundaries one can construct quantities proportional to the determinants which are invariant under Weyl transformations as well as reparameterization of the surface.[1] The techniques used in the derivation are adapted from those used to study Weyl invariance and fix the critical dimension in Polyakov's path-integral formulation of string theory [1, 2, 3].

To show the utility of these invariants, they are used to obtain explicit representations of the determinants of the scalar Laplacian for both Dirichlet and Neumann boundary conditions on the flat disk and the flat annulus. Zeta function regularization is used to define the formally divergent expressions for the determinants [4].

To write the invariants on a Riemann surface, M, with smooth boundaries, ∂M, we use the following notation. The scalar Laplacian is

$$\Delta = -(1/\sqrt{g})\partial_a(\sqrt{g}\, g^{ab}\partial_b). \tag{1}$$

[1] The method is also applicable to Laplace operators acting on higher rank tensors