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A Mathematical Classification of the One-Dimensional Deterministic Cellular Automata

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Abstract. We propose a theoretical classification of one-dimensional deterministic cellular automata in two types, type S and type O. This classification is connected with the phenomenological classification of S. Wolfram.

1. Introduction

This paper is devoted to theoretical investigations concerning the classification of one-dimensional deterministic cellular automata [1, 2]. Such an automaton is described (see in Sect. 2 for more precise definitions) by an evolution of the form $x_i(t+1) = f(x_{i-r_L}(t), x_{i-r_L+1}(t), \ldots, x_{i+r_R}(t))$, where for each lattice site $i \in \mathbb{Z}$ and discrete time $t, x_i(t)$ belongs to a fixed finite set E, (e.g. $\mathbb{Z}/p\mathbb{Z}$).

Among all the possible properties of an automaton which one may investigate, there is a natural one which we call surjectivity. We say that an automaton is surjective if for any finite configuration $(y_i, y_{i+1}, \ldots, y_{i+n})$ and any time $t > 0$, one can find initial conditions $(x_j(0))$ such that $x_{i+k}(t) = y_{i+k}$ for $k \in \{0, \ldots, n\}$. It is obvious that an automaton is surjective if and only if any finite configuration has at least one antecedent. The main result of this paper is that a deterministic one-dimensional automaton is surjective if and only if all finite configurations have the same number of antecedents. This result is typical of one-dimensional automata and means that, for such systems, surjective is equivalent to equiprobability of finite configurations, given equiprobable initial conditions.

We thus classify automata in two types: surjective automata will be said of type S, the other ones of type O.

Let us comment on the connection between this classification and the one introduced by Wolfram: In [3], S. Wolfram gave a phenomenological classification of such automata based on computer experiments. Class I consists of automata such that for $t \geqq T$ the values $x_i(t)$ do not depend on the initial conditions $x_j(0)$. Class II consists of automata such that for $t \geqq T$, $x_i(t)$ only depends on the initial values $x_j(0)$ at a finite number, say $m+1$, of adjacent sites with m independent of t, i.e. $x_i(t) =$