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Conditional Transformation of Drift Formula and Potential Theory for $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$

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Abstract. Using conditional Brownian motion and the transformation of drift formula (of Cameron–Martin, Girsarov, Maruyama) we give integral conditions on a vector field b which imply the harmonic measures and Green functions for $\frac{1}{2}\Delta$ and $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$ on a bounded Lipschitz domain D are equivalent. By equivalent we mean there exist two-sided inequalities with constants depending only on b and D. This enables one to conclude the potential theory for $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$ on D and $\frac{1}{2}\Delta$ on D are the same.

1. Introduction

The purpose of this paper is to study the operator $L = \frac{1}{2}\Delta + b(\cdot)\cdot\nabla$ on a domain D. We shall impose integral conditions on b which allow b to have singularities and D will be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geqq 2$. Under our conditions on b and D, there will be two-sided inequalities $c^{-1}G(x,y) \leqq G_L(x,y) \leqq cG(x,y)$ and $c^{-1}w^x(dz) \leqq w_L^x(dz) \leqq cw^x(dz)$ between Green functions and harmonic measures for L and $\frac{1}{2}\Delta$ on D. The approach is probabilistic and follows closely Cranston, Fabes and Zhao (1986). The ideas here can trace their history to the works of Chung (1985), Falkner (1983) and Zhao (1983, 1984).

There are two main differences between treating $\frac{1}{2}\Delta + q$ as in the first work mentioned above and L in the present work. The first is in the use of the transformation of drift formula (sometimes called the Cameron–Martin–Girsanov formula which was also studied by Maruyama (1954)) instead of the Feynman–Kac formula. The second difference is in the use of the stochastic process version of the John–Nirenberg Theorem (see Delacherie–Meyer (1980)) instead of Khasminski's Lemma. In the previous works q was taken in the Kato class K_d^{loc} (see the proof for Corollary 3.14). The condition that arises from our techniques is that $|b|^2 \in K_d^{\mathrm{loc}}$ and $|b| \in K_{d+1}^{\mathrm{loc}}$.

The two-sided inequalities between Green functions and harmonic measures enable one to obtain potential theoretic results for L which are known to hold for $\frac{1}{2}\Delta$ and we give several though by no means all consequences that may be