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Index of a Family of Dirac Operators on Loop Space*

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Dedicated to Walter Thirring on his 60th birthday

Abstract. We use methods of constructive field theory to generalize index theory to an infinite-dimensional setting. We study a family of Dirac operators Q on loop space. These operators arise in the context of supersymmetric non-linear quantum field models with Hamiltonians $H=Q^2$. In these models Q is self-adjoint and Fredholm. A natural grading operator Γ exists such that $\Gamma Q+Q\Gamma=0$. We study $Q_+=P_-QP_+$, where $P_\pm=\frac{1}{2}(1\pm\Gamma)$ are the orthogonal projections onto the eigenspaces of Γ. We calculate the index $i(Q_+)$ for Wess-Zumino models defined by a superpotential $V(\varphi)$. Here V is a polynomial of degree $n\geqq 2$. We establish that $i(Q_+)=n-1=\deg\partial V$. In particular, the field theory models have unbroken supersymmetry, and (for $n\geqq 3$) they have degenerate vacua. We believe that this is the first index theorem for a Dirac operator that couples infinitely many degrees of freedom.

I. Introduction

In this paper we present index theory for a family of Dirac operators on loop space. Since loop space is infinite-dimensional, the mathematical framework requires careful analysis. Each Dirac operator Q which we study will be associated with a stochastic process over loop space. The most interesting such processes are non-Gaussian. Our mathematical presentation relies on methods of constructive quantum field theory [1] to define and study the infinite-dimensional processes. We proceed by several steps:

1. We define a family of Dirac operators Q and appeal to a companion paper for mathematical existence theorems [2].

2. For each such Q, we introduce a family $Q(\kappa)$, $0\leqq\kappa\leqq\infty$, which interpolates between $Q\equiv Q(\infty)$ and $Q(0)=Q_0+Q_{i,0}$. Here Q_0 is associated with a Gaussian

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