Commun. Math. Phys. 112, 3–20 (1987)

Communications in Mathematical Physics

Some Rigorous Results on the Sherrington-Kirkpatrick Spin Glass Model

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Dedicated to Walter Thirring on his 60th birthday

Abstract. We prove that in the high temperature regime ($T/J > 1$) the deviation of the total free energy of the Sherrington-Kirkpatrick (S-K) spin glass model from the easily computed $\log \mathrm{Av}(Z_N(\{\beta J\}))$ converges in distribution, as $N \to \infty$, to a (shifted) Gaussian variable. Some weak results about the low temperature regime are also obtained.

1. Introduction

We consider the Sherrington-Kirkpatrick [1] spin glass model, with the mean field Hamiltonian

$$H = - \sum_{1 \leq i < j \leq N} \frac{J_{ij}}{\sqrt{N}} \sigma_i \sigma_j , \tag{1.1}$$

where the spins $\sigma_1, \ldots, \sigma_N$ take values ± 1, and the J_{ij}'s are independent identically distributed random variables with mean zero and variance J^2. (The randomness is reflected in the unusual scaling factor $1/\sqrt{N}$ in H.) It is believed that this model has trivial behavior at high temperatures (for $\beta J < 1$) while at low temperatures ($\beta J > 1$) it has a rich structure of "Gibbs states" or "valleys" described by Parisi's Replica Symmetry Breaking solution; see [2] for a review of the history, the current status, and an extended list of references – on this and related subjects.

While the Parisi solution is widely believed to be exact, it is not claimed to be rigorous. In fact, we have not found in the literature a complete treatment of even the high temperature region. The main purpose of this note is to provide a full description of the free energy in that regime. We supplement the existing analysis by deriving the limiting probability law of the *fluctuations* in the total free energy. In addition, the results of Sherrington and Kirkpatrick [1] and Thouless et al. [3]

* Research supported in part by the NSF grant PHY-8605164. Present address: Courant Institute of Mathematical Sciences, 251 Mercer St., New York, NY 10012, USA

** Research supported in part by the NSF grant DMR 86-12369

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