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The Spectrum of a Quasiperiodic Schrödinger Operator

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Abstract. The spectrum $\sigma(H)$ of the tight binding Fibonacci Hamiltonian ($H_{mn} = \delta_{m,n+1} + \delta_{m+1,n} + \delta_{m,n}\mu v(n)$, $v(n) = \chi_{[-\omega^3,\omega^2[}((n-1)\omega)$, $1/\omega$ is the golden number) is shown to coincide with the dynamical spectrum, the set on which an infinite subsequence of traces of transfer matrices is bounded. The point spectrum is absent for any μ, and $\sigma(H)$ is a Cantor set for $|\mu| \geqq 4$. Combining this with Casdagli's earlier result, one finds that the spectrum is singular continuous for $|\mu| \geqq 16$.

Consider the discrete Schrödinger operator H acting on doubly infinite sequences $(\ldots, \psi(-1), \psi(0), \psi(1), \ldots)$, and defined by $(H\psi)(n) = \psi(n+1) + \psi(n-1) + \mu v(n)\psi(n)$ with the potential

$$v(n) = \chi_{[-\omega^3,\omega^2[}((n-1)\omega). \tag{1}$$

Here $\omega = (\sqrt{5}-1)/2$, and χ_I is the characteristic function of the interval I. H is a bounded self-adjoint operator on $l^2(\mathbb{Z})$; we are interested in its spectrum. This problem was originally proposed by Kohmoto et al., [1] and Ostlund et al., [2]. Mathematical properties of the sequence (1) were discussed earlier [3], and $v(n)$ appeared also in some models of dissipative systems [4, 5]. The interest in this particular, nongeneric example of a quasiperiodic Schrödinger operator is explained by its connection with a simple dynamical system whose evolution can be studied with relative ease, so that one may hope for detailed numerical and rigorous results. Moreover, the spectrum of H has long been suspected to be singular continuous, irrespectively of the value of μ.

Let $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$. For any solution ψ of $H\psi = E\psi$ one can write

$$\Psi_N = T_N T_{N-1} \ldots T_1 \Psi_0, \quad N \geqq 1,$$

where

$$\Psi_N = \begin{pmatrix} \psi(N+1) \\ \psi(N) \end{pmatrix}, \quad T_N = \begin{pmatrix} E - \mu v(N) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

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