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Analytics of Period Doubling

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Abstract. Approximate analytic solutions for periodic orbits of the quadratic map $x \to rx(1-x)$ are developed using algebraic methods. These solutions form the basis of an exact algorithm which predicts the quantitative order of periodic points characteristic of the Feigenbaum scenario. The algorithm holds for any one dimensional unimodal map. A general procedure is developed which permits calculation of period doubling parameters for large period orbits from those of low period to any desired degree of accuracy. Explicit equations are given through second order.

1. Introduction

The emergence of periodic points due to subharmonic bifurcations displayed by the prototypic quadratic map [1–3],

$$x \to F_0(x) \equiv rx(1-x), \tag{1}$$

can be traced analytically using algebraic methods. The procedure to be developed is implied by the approximate renormalization method of Helleman [4–6] who was concerned with scaling. The present study is determination of periodic point structure implied by scaling. These points arise as follows: as r progresses from $r_2 = 3$ a period 2 orbit which had evolved from period 1 will subsequently evolve at $r_4 = 3.44\ldots$ to a period 4 harmonic of F_0. The period 4 harmonic are four periodic points of period 2 for F_0^2. At $r_8 = 3.54\ldots$ the period 4 harmonic evolves to the period 8 harmonic of F_0 which are points of period 2 for F_0^4. This process of period doubling continues as r increases until the accumulation point $r_\infty = 3.5699\ldots$ at which appears the harmonic of infinite period. This is a particular example of the Feigenbaum period doubling scenario displayed quite generally by one-dimensional unimodal maps. The harmonic of period p, or p-cycle, is the orbit of p