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Towards a Unified Dynamical Theory of the Brownian Particle in an Ideal Gas

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Abstract. We consider the trajectory $Q^M(t)$ of a Brownian particle of mass M in an ideal gas of identical particles of mass 1 and of density 1 in equilibrium at inverse temperature 1 (the dynamics is uniform motion plus elastic collisions with the Brownian particle). Our theory, in dimension one, describes a variety of limiting processes – containing the Wiener process and the Ornstein-Uhlenbeck process – for $A^{-1/2}\,Q^{M(A)}(At)$ depending on the asymptotic behaviour of $M(A)$. Part of the theory is hypothetical while another part relies upon known results. We also prove that, if $A^{\frac{1}{2}+\varepsilon} \ll M(A) \ll A$, then $A^{-1/2}\,Q^{M(A)}(At)$ converges to a Wiener process whose variance is known from papers of Sinai-Soloveichik and of the present authors.

1. Introduction

Ed Nelson's classical notes about Brownian motion, N (1967), also containing an exciting historical account, stressed the necessity to derive Brownian motion from Hamiltonian principles. "The problem, or one formulation of it, is to deduce each of the following theories from the one below it:

Einstein–Smoluchowski
Ornstein–Uhlenbeck
Maxwell–Boltzmann
Hamilton–Jacabi."

His notes, in fact, show that "the Einstein-Smoluchowski theory is in a rigorous and strong sense the limiting theory of the Ornstein-Uhlenbeck theory." (Note that the mathematical model of the first theory is the Wiener process.)

The aim of the present paper is to realize the program for a Brownian particle interacting with an ideal gas of point particles. A rough outline of our theory is the following: if we start from a Gibbs equilibrium state, then the model contains a functional parameter describing the interdependence between the mass ratio of the Brownian particle and of the gas particles on one side and the space-time scaling of