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Complete Integrability of Relativistic Calogero-Moser Systems and Elliptic Function Identities

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Abstract. Poincaré-invariant generalizations of the Galilei-invariant Calogero-Moser N-particle systems are studied. A quantization of the classical integrals $S_1, \ldots, S_N$ is presented such that the operators $\hat{S}_1, \ldots, \hat{S}_N$ mutually commute. As a corollary it follows that $S_1, \ldots, S_N$ Poisson commute. These results hinge on functional equations satisfied by the Weierstrass σ- and $\mathscr{P}$-functions. A generalized Cauchy identity involving the σ-function leads to an $N \times N$ matrix L whose symmetric functions are proportional to $S_1, \ldots, S_N$.

1. Introduction

Recently, new integrable classical N-particle systems have been discovered [1] that may be viewed as relativistic generalizations of the well-known nonrelativistic Calogero-Moser systems [2]. The time translation, space translation, and boost generators of these systems are given by

$$H = mc^2 \sum_{i=1}^{N} \operatorname{ch}\theta_i \prod_{j \neq i} f(q_i - q_j), \tag{1.1}$$

$$P = mc \sum_{i=1}^{N} \operatorname{sh}\theta_i \prod_{j \neq i} f(q_i - q_j), \tag{1.2}$$

$$B = -\frac{1}{c} \sum_{i=1}^{N} q_i. \tag{1.3}$$

Here, m denotes the particle mass, c the speed of light, θ the particle rapidity, and q the canonically conjugate generalized position. Moreover, the potential energy function $f(q)$ reads

$$f(q) = (a + b\mathscr{P}(q))^{1/2}, \tag{1.4}$$

where a and b are arbitrary constants and where $\mathscr{P}$ is the Weierstrass $\mathscr{P}$-function. This choice of f not only guarantees Poincaré invariance, but also the existence of N independent integrals for the H flow, given by

$$S_k = \sum_{\substack{I \subset \{1, \ldots, N\} \\ |I| = k}} \exp\left(\sum_{i \in I} \theta_i\right) \prod_{\substack{i \in I \\ j \notin I}} f(q_i - q_j), \qquad k = 1, \ldots, N. \tag{1.5}$$

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