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Elliptic Genera and Quantum Field Theory

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Abstract. It is shown that in elliptic cohomology – as recently formulated in the mathematical literature – the supercharge of the supersymmetric nonlinear sigma model plays a role similar to the role of the Dirac operator in K-theory. This leads to several insights concerning both elliptic cohomology and string theory. Some of the relevant calculations have been done previously by Schellekens and Warner in a different context.

If M is a spin manifold of dimension n, we can consider the Dirac operator $i\not{D}$, acting on a field ψ_α which is a section of the spinor bundle S. More generally, if R is any representation of the structure group $\mathrm{Spin}(n)$ of the tangent bundle, we can consider the Dirac operator acting on a field $\psi_{\alpha i}$, α, and i being respectively a spinor index and an index labeling the representation R; in mathematical terms, ψ is a section of $S\otimes T_R$, T_R being the $\mathrm{Spin}(n)$ bundle associated with the representation R of $\mathrm{Spin}(n)$.

In [1], an infinite series of representations R_i, $i=0,1,2,\ldots$ was singled out. The first few were

$$\begin{aligned} R_0&=1\,,\\ R_1&=T,\\ R_2&=\Lambda^2T\oplus T,\\ R_3&=\Lambda^3T\oplus(T\otimes T)\oplus T. \end{aligned}\tag{1}$$

Here 1 is the trivial representation, T is the fundamental (vector) representation of $SO(N)$, and Λ^k denotes the k^{th} antisymmetric tensor product. The special role of these operators was as follows. Let M be a spin manifold with a compact symmetry group G. It is sufficient in what follows to consider an S^1 [i.e., $U(1)$] subgroup of G. Let K be the generator of this S^1 action. Assuming that the symmetry generated by K lifts to the spinor bundle, K commutes with the Dirac operator $i\not{D}$ (or a

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