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Local and Non-Local Conserved Quantities for Generalized Non-Linear Schrödinger Equations

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Abstract. It is shown how to construct infinitely many conserved quantities for the classical non-linear Schrödinger equation associated with an arbitrary Hermitian symmetric space G/K. These quantities are non-local in general, but include a series of local quantities as a special case. Their Poisson bracket algebra is studied, and is found to be a realization of the "half" Kac-Moody algebra $\mathscr{k}_R \otimes \mathbb{C}[\lambda]$, consisting of polynomials in positive powers of a complex parameter λ which have coefficients in the compact real form of $\mathscr{k}$ (the Lie algebra of K).

1. Introduction

Fordy and Kulish [1] have considered a class of non-linear partial differential equations, each associated with an Hermitian symmetric space G/K, which are of the form

$$iq_t^\alpha = q_{xx}^\alpha - q^\beta q^\gamma q^{\delta *} R^\alpha_{\beta\gamma-\delta}\,, \tag{1.1}$$

where summation is implied over repeated indices. $q^\alpha(x,t)$ are fields in one space dimension whose label α denotes a root of $\mathscr{g}$ (the Lie algebra of G) such that the step operator e_α does not lie in $\mathscr{k}$ (the Lie algebra of K). R is the "curvature tensor" defined by

$$e_\alpha R^\alpha_{\beta\gamma-\delta} = [e_\beta[e_\gamma, e_{-\delta}]]\,. \tag{1.2}$$

A special case of (1.1), corresponding to $G=SU(2)$, is the non-linear Schrödinger (NLS) equation

$$iq_t = q_{xx} + 2|q|^2 q\,. \tag{1.3}$$

Equation (1.1) will be referred to as the Generalized non-linear Schrödinger (GNLS) equation associated with G/K. The NLS equation is known to have infinitely many conserved quantities which are local [in the sense that the currents are expressed only in terms of the fields $q(x,t)$, $q^*(x,t)$ and their derivatives at a point], and are in involution (i.e. their Poisson bracket algebra is abelian). The aim