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Inverse Scattering for the Heat Operator and Evolutions in 2 + 1 Variables

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Abstract. The asymptotic behavior of functions in the kernel of the perturbed heat operator $\partial_1^2 - \partial_2 - u(x)$ suffice to determine $u(x)$. An explicit formula is derived using the $\bar{\partial}$ method of inverse scattering, complete with estimates for small and moderately regular potentials u. If u evolves so as to satisfy the Kadomtsev–Petviashvili (KP II) equation, the asymptotic data evolve linearly and boundedly. Thus the KP II equation has solutions bounded for all time. A method for calculating nonlinear evolutions related to KP II is presented. The related evolutions include the so-called "KP II Hierarchy" and many others.

I. Introduction

Let $u = u(x)$ denote both a function of $x \in \mathbb{R}^2$ and the operation of multiplication by that function. This work investigates the inverse scattering problem for the differential operator

$$\partial_1^2 - \partial_2 - u. \tag{I.1}$$

There are two main results. First, for small smooth $u \in L^1 \cap L^2(\mathbb{R}^2)$, the operator is determined by the leading coefficients of asymptotically exponential functions in its kernel. That theorem has several parts. Let $z \in \mathbb{C}$, write $v = v(z) = (z, z^2) \in \mathbb{C}^2$, and let $m(x, z)$ be a bounded function such that $\psi = e^{x \cdot v} m(x, z)$ is in the kernel of (I.1). Then for u small in $L^1 \cap L^2(\mathbb{R}^2)$ there is a unique such m satisfying $m(x, z) \to 1$ as $|x| \to \infty$ (Theorem 1.II). Hence ψ is asymptotic to $e^{x \cdot v}$. If u has some decay as $|x| \to \infty$, then m has the asymptotic behavior

$$m = 1 + \frac{\beta(z)}{x_1 + 2zx_2} + \frac{\alpha(z) e^{x(\bar{v} - v)}}{x_1 + 2\bar{z}x_2} + o\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty \quad \text{for } \operatorname{Im} z \neq 0. \tag{I.2}$$

Here α and β are bounded functions, with

$$\alpha(z) = \frac{1}{2\pi i} \operatorname{sgn}(\operatorname{Im} z) \int_{\mathbb{R}^2} e^{x \cdot (v - \bar{v})} u(x) m(x, z) dx \tag{I.3}$$

(Theorem 4.II).