Commun. Math. Phys. 108, 25–40 (1987)

Communications in Mathematical Physics

The Schrödinger Equation and Canonical Perturbation Theory

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Abstract. Let $T_0(\hbar, \omega) + \varepsilon V$ be the Schrödinger operator corresponding to the classical Hamiltonian $H_0(\omega) + \varepsilon V$, where $H_0(\omega)$ is the d-dimensional harmonic oscillator with non-resonant frequencies $\omega = (\omega_1, \dots, \omega_d)$ and the potential $V(q_1, \dots, q_d)$ is an entire function of order $(d+1)^{-1}$. We prove that the algorithm of classical, canonical perturbation theory can be applied to the Schrödinger equation in the Bargmann representation. As a consequence, each term of the Rayleigh–Schrödinger series near any eigenvalue of $T_0(\hbar, \omega)$ admits a convergent expansion in powers of $\hbar$ of initial point the corresponding term of the classical Birkhoff expansion. Moreover if V is an even polynomial, the above result and the KAM theorem show that all eigenvalues $\lambda_n(\hbar, \varepsilon)$ of $T_0 + \varepsilon V$ such that $n\hbar$ coincides with a KAM torus are given, up to order ε^∞, by a quantization formula which reduces to the Bohr–Sommerfeld one up to first order terms in $\hbar$.

I. Introduction and Statement of Results

Consider the formal Schrödinger operator acting in $L^2(\mathbb{R}^d)$:

$$T(\hbar, \varepsilon) = T_0(\hbar) + \varepsilon V. \tag{1.1}$$

Here $q \equiv (q_1 \cdots q_d) \in \mathbb{R}^d$, $q \to V(q)$ is a real-valued function, and ε is a non-negative number. The operator $T(\hbar, \varepsilon)$ is obtained through formal quantization (i.e., through the replacement $p_i \to i\hbar(\partial/\partial q_i)$) of the classical Hamiltonian defined on $\mathbb{R}^{2d}$

$$H(p, q; \varepsilon) = H_0(p, q) + \varepsilon V(q), \quad p \equiv (p_1 \cdots p_d) \in \mathbb{R}^d, \quad \{p_i, q_j\} = \delta_{ij} \tag{1.2}$$

Let $H_0(p, q)$ be canonically integrable over $\mathbb{R}^{2d}$, namely (see e.g. [4, p. 289]) let $(\mathbb{R}^2 \backslash \{0\})^d$ be canonically foliated into $(\mathbb{R}_+)^d \times \mathbb{T}^d$ through globally defined action-angle variable $(A, \phi) = C(p, q)$, $A \in \mathbb{R}^d_+$, $\phi \in \mathbb{T}^d$, C being a completely canonical map of $(\mathbb{R}^2 \backslash \{0\})^d$ onto $\mathbb{R}^d_+ \times \mathbb{T}^d$ such that $H_0(C^{-1}(A, \phi)) \equiv f_0(A)$. Accordingly, we rewrite (1.2) in the canonically equivalent form

$$H(C^{-1}(A, \phi), \varepsilon) = f_0(A) + \varepsilon V(A, \phi), \quad V(A, \phi) \equiv V(C^{-1}(A, \phi)). \tag{1.3}$$