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On a Conformally Invariant Elliptic Equation on R^n

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Abstract. For $n \geqq 3$, the equation $\Delta u + |u|^{4/(n-2)}u = 0$ on $\mathbb{R}^n$ has infinitely many distinct solutions with finite energy and which change sign.

In [4], Gidas-Ni-Nirenberg proved that any positive solution of the elliptic equation

$$\Delta u + |u|^{4/(n-2)}u = 0, \quad u \in C^2(R^n), \quad n \geqq 3, \tag{1}$$

which has finite energy, namely

$$\int_{R^n} |\nabla u|^2 dx < \infty, \tag{2}$$

is necessarily of the form

$$u(x) = \left(\frac{\sqrt{n(n-2)}\,a}{a^2 + |x-\xi|^2}\right)^{(n-2)/2}, \tag{3}$$

where $a > 0$, $\xi \in R^n$. Thereafter, some people tried to show without success that all the solutions of the problem (1)–(2), which are positive somewhere, are given by (3). Their efforts have to be in vain, as we will see shortly that the problem actually has a lot of solutions other than those given by (3). Our main result in this note can be stated as follows.

Theorem. *There exists a sequence of solutions u_k of* (1)–(2), *such that* $\int_{R^n} |\nabla u_k|^2 dx \to \infty$ *as* $k \to \infty$.

We remark that Eq. (1) is invariant under the conformal transformations of R^n. Thus, if $u(x)$ is a solution, then for any $\lambda > 0$ and $\xi \in R^n$, $\lambda^{(n-2)/2} u[(x-\xi)/\lambda]$ is also a solution. Moreover, all solutions obtained in this way have the same energy, and we will say that these solutions are equivalent. In particular, the solutions (3) are equivalent. Our theorem implies the existence of infinitely many inequivalent solutions to the problem (1)–(2).