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Remarks on J. Langer and D. A. Singer Decomposition Theorem for Diffeomorphisms of the Circle

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Abstract. We give a simple proof that all C^4 diffeomorphisms of the torus can be factorized into a finite number of diffeomorphisms commuting with reflection.

In one dimension, C^3 suffices and even C^2 can yield that the factors are almost diffeomorphisms. (The derivatives of the function and the inverse are in L^1 and are positive.)

In one dimension under C^∞ assumptions, this had been proved by J. Langer and D. A. Singer in their study of geodesic fields by different methods.

1. Introduction

J. Langer and D. A. Singer proved in [1] that $\mathrm{Diff}_+(S^1)$—the group of C^∞ orientation preserving diffeomorphisms of the circle—is generated by symmetric diffeomorphisms; that is, every C^∞ diffeomorphism is the composition of a finite number of diffeomorphisms that commute with reflections. (Taking $S = \mathbb{R}/\mathbb{Z}$, a reflection about θ is $R_\theta(x) = -(x-\theta)+\theta \pmod 1$, $f\colon S^1 \to S^1$ commutes with reflections if $f \circ R_\theta = R_\theta \circ f$.)

The interest of this theorem for the authors above was that, in the same paper, they showed it implies that given any orientation preserving C^∞ diffeomorphism f, there is a C^∞ gradient flow in the annulus that sends the point θ in the inner boundary to $f(\theta)$ in the outer boundary; this in turn, can be used to characterize which transformations in rays can be achieved through geodesic flows in conformal metrics; in more physical terms, they characterize the transformations that can be achieved by isotropic lenses (scalar index of refraction) in 2-D. Other applications are also possible. Since the conformal group in two dimensions (indefinite metric) is just the product of two diffeomorphism groups of the circle, it suffices to show invariance under changes that commute with reflections to show conformal invariance.

The object of this note is to provide a different and simpler proof of the result that works with weaker differentiability assumptions and obtains stronger differentiability in the factors.