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Existence and Uniqueness of Positive Solutions of Semilinear Elliptic Equations with Coulomb Potentials on $\mathbb{R}^3$

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Dedicated to the memory of Professor Mark Kac

Abstract. We study the positive solutions of a semilinear equation with a Coulomb potential on $\mathbb{R}^3$. We give a new uniqueness theorem for the positive radial solutions of such an equation and we apply these results to the Thomas-Fermi-Dirac-von Weizsäcker equation without electrostatic repulsion.

1. Introduction

In this article we shall discuss the existence and uniqueness of positive classical solutions $u(x)$ of the problem

$$-\Delta u - Z|x|^{-1}u + a(u) = 0 \quad \text{in} \quad \mathbb{R}^3, \qquad u(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{1}$$

In Eq. (1), Z is a positive real constant and the function $a(\cdot)$ satisfies the following hypothesis:

A1. $a(t)$ is locally Lipschitz continuous for $t \geqq 0$, $a(0) = 0$ and

$$\lim_{t \downarrow 0} (a(t)/t) \equiv \lambda > 0. \tag{2}$$

A2. There are positive constants δ, C_+ and C_- such that

$$C_- t^p \leqq a(t) \leqq C_+ t^p \tag{3}$$

for all $t \geqq \delta$. Here, $1 < p < \infty$.

A3. For all $t > 0$,

$$F(t) \equiv 2 \int_0^t a(s)\,ds > 0. \tag{4}$$

Let us also define

$$\alpha \equiv \inf_{t > 0} (F(t)/t^2); \tag{5}$$

the hypothesis A1, A2, and A3 ensure that $0 < \alpha \leqq \lambda$.

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