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# Stability of Coulomb Systems with Magnetic Fields

## III. Zero Energy Bound States of the Pauli Operator

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**Abstract.** It is shown that there exist magnetic fields of finite self energy for which the operator $\sigma\cdot(p-A)$ has a zero energy bound state. This has the consequence that single electron atoms, as treated recently by Fröhlich, Lieb, and Loss [1], collapse when the nuclear charge number $z \geqq 9\pi^2/8\alpha^2$ ($\alpha$ is the fine structure constant).

## I. Introduction

In an accompanying paper [1] the stability of the hydrogen atom in magnetic fields is studied. The authors considered the following Hamiltonian

$$H=[\sigma\cdot(p-A)]^2-z/|x| \tag{1.1}$$

whose ground state energy was denoted by $E_0(B,z)$. Here the $\sigma_i$'s are the Pauli matrices and $A$ is the vector potential, $B=\operatorname{curl}A$. $H$ acts on 2-component spinors $\psi$. In particular, it was shown that there is a critical number $z_c>0$ such that $E(z) = \inf_B (E_0(B,z)+\varepsilon\int B^2)$ was finite whenever $z<z_c$ and $E(z)=-\infty$ for $z>z_c$. $\varepsilon=(8\pi\alpha^2)^{-1}$ and $\alpha$ is the fine structure constant $\simeq(137.04)^{-1}$. For the physical interpretation of these results see [1]. When they first did their work, the authors did not know whether $z_c$ was finite or not. However they show, among other results, that a necessary and sufficient condition for the finiteness of $z_c$, is that the equation

$$\sigma\cdot(p-A)\psi=0\,, \tag{1.2}$$

is valid for some $A$ and some $\psi$, which satisfy

$$\psi\in H^1(\mathbb{R}^3), \quad \text{i.e.} \quad \psi, \nabla\psi\in L^2(\mathbb{R}^3), \tag{1.3a}$$

$$A\in L^6(\mathbb{R}^3), \quad \operatorname{div}A=0 \quad \text{and} \quad B\equiv\operatorname{curl}A\in L^2(\mathbb{R}^3). \tag{1.3b}$$

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