Commun. Math. Phys. 104, 251–270 (1986)

Communications in
Mathematical
Physics

Stability of Coulomb Systems with Magnetic Fields

I. The One-Electron Atom

Jürg Fröhlich[1,*], Elliott H. Lieb[2,**], and Michael Loss[2,***]

[1] Theoretical Physics, ETH-Hönggerberg, CH-8093 Zürich, Switzerland
[2] Departments of Mathematics and Physics, Princeton University, Jadwin Hall, P.O.B. 708, Princeton, NJ 08544 USA

Abstract. The ground state energy of an atom in the presence of an external magnetic field B (with the electron spin-field interaction included) can be arbitrarily negative when B is arbitrarily large. We inquire whether stability can be restored by adding the self energy of the field, $\int B^2$. For a hydrogenic like atom we prove that there is a critical nuclear charge, z_c, such that the atom is stable for $z<z_c$ and unstable for $z>z_c$.

1. Introduction

The problem of the stability of an atom (i.e. the finiteness of its ground state energy) was solved by the introduction of the Schrödinger equation in 1926. While it is true that Schrödinger mechanics nicely takes care of the $-ze^2/r$ Coulomb singularity at $r=0$ (here $z|e|$ is the nuclear charge), a more subtle problem that has to be considered is the interaction of the atom with an external magnetic field $B(x)$ with vector potential $A(x)$ and $B=\operatorname{curl} A$. In this paper the problem of the one-electron atom in a magnetic field is studied; in a subsequent paper [6] some aspects of the many-electron and many-nucleus problem will be addressed.

Units. Our unit of length will be half the Bohr radius, namely $l=\hbar^2/(2me^2)$. The unit of energy will be 4 Rydbergs, namely $2me^4/\hbar^2=2mc^2\alpha^2$, where α is the fine structure constant $e^2/(\hbar c)$. The magnetic field B is in units of $|e|/(l^2\alpha)$. The vector potential satisfies $B=\operatorname{curl} A$. The magnetic field energy ($\int B^2/8\pi$) is, in these units,

$$\varepsilon \int B^2\,, \qquad 1/\varepsilon=8\pi\alpha^2\,. \tag{1.1}$$

---

* Work partially supported by U.S. National Science Foundation grant DMS-8405264 during the author's stay at the Institute for Advanced Study, Princeton, NJ, USA
** Work partially supported by U.S. National Science Foundation grant PHY-8116101-A03
*** Work partially supported by U.S. and Swiss National Science Foundation Cooperative Science Program INT-8503858. Current address: Institut f. Mathematik, FU Berlin, Arnimallee 3, D-1000 Berlin 33