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Integrable Cases of a Rigid Body Dynamics and Integrable Systems on the Ellipsoids

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Abstract. Infinite-dimensional sets of integrable cases are found for the equations of a rigid body rotation around a fixed point in an axially symmetric potential field and also in more complicated fields in the presence of some symmetry of the rigid body inertia tensor.

1. Introduction

Rotation of a rigid body around a fixed point in some potential field is described by the Lagrangian system determined on the Lie group SO(3) with the Lagrangian

$$L=\tfrac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)-V(Q_j^i). \tag{1.1}$$

Here I_k are the eigenvalues of the inertia tensor, ω is the angular velocity vector, with components ω_k, Q is the orthogonal matrix, with elements Q_j^i, determining the position of the rigid body, $V(Q_j^i)$ is a potential function. The classical integrable cases of a rigid body dynamics [1] have the following common properties: firstly all of them depend on some finite and not great number of parameters, determining the special forms of the potential function $V(Q_j^i)$ and special values of the inertia tensor components I_k, and secondly the dynamics of trajectories of the corresponding integrable Hamiltonian systems is a linear winding of two-dimensional invariant tori $\mathbb{T}^2$.

The present work is devoted to construction of new cases of integrability for the equations of a rigid body dynamics, depending on an infinite number of parameters which determine the form of the potential function $V(Q_j^i)$. A number of geometrical methods are used connected with the existence of the maps

$$S^3 \xrightarrow{f} \mathrm{SO}(3) \xrightarrow{g} S^2. \tag{1.2}$$

The map f is the universal covering, the map g is the fibration, its fibres are circumferences S^1; S^n is an n-dimensional sphere. The composition of the maps $g\circ f$ is the known Hopf fibration. Due to the presence of the maps f and g (1.2), it is