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Comments

On the Concept of Attractor: Correction and Remarks

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The following consists of three unrelated comments on the author's paper [1].

1. Correction

Let f be a continuous map from a compact metric space X to itself, with n^{th} iterate denoted by f^n, and let $A \subset X$ be a closed non-vacuous subset with $f(A) = A$. Consider the following two properties of A.

(I) *For any sufficiently small neighborhood U of A, the intersection of the images $f^n(U)$ for $n \geqq 0$ is equal to A (compare Smale* [2, p. 786]*).*

(II) (Asymptotic stability) *For any sufficiently small neighborhood U, the successive images $f^n(U)$ converge to A, in the sense that for any neighborhood V there exists n_0 so that $f^n(U) \subset V$, for $n \geqq n_0$.*

In [1, Sect. 1] the author mistakenly described an example satisfying (I) but not (II). (The example was based on a remark of Besicovitch [3], which was corrected in a later paper [4].) In fact, (I) implies (II). The following proof is a minor modification of Hurley [8, Lemma 1.6], which demonstrates a corresponding statement for flows on a compact manifold. The proof shows also that (I) implies the existence of arbitrarily small neighborhoods $W \supset A$ with $f(W) \subset W$.

Proof that (I) *implies* (II). Let U be an open neighborhood which is small enough so that the intersection of the forward images of the closure $\bar{U}$ is equal to A. Let U_n be the open neighborhood consisting of all points x such that $f^i(x) \in U$ for $0 \leqq i \leqq n$. Thus $U = U_0 \supset U_1 \supset \ldots \supset A$ and $f(U_n) \subset U_{n-1}$. Hence the intersection W of the U_n satisfies $f(W) \subset W$. We will show that W is equal to U_n for n sufficiently large, and hence that W is an open set. Otherwise, for infinitely many integers n there must exist a point x_n which belongs to U_n but not U_{n+1}. Let $y_n = f^n(x_n) \in U$. Then we can choose some subsequence of these points y_n which converges to a point $y \in \bar{U}$. Since y_n belongs to the intersection of the sets $f^i(\bar{U})$ for $0 \leqq i \leqq n$, it follows that y belongs to the intersection of all of the $f^i(\bar{U})$, which is equal to A by hypothesis. But $f(y_n) \notin U$, hence $f(y) \notin U$, contradicting the hypothesis that $f(A) = A \subset U$. This proves that W is open. Hence the compact set $\bar{W} \subset \bar{U}$ is a neighborhood of A with $f(\bar{W}) \subset \bar{W}$. It follows easily from compactness that the successive images $\bar{W} \supset f(\bar{W}) \supset f^2(\bar{W}) \supset \ldots$ with intersection A actually converge to A in the sense described in (II). □