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Quantum Logics and Convex Geometry

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Abstract. The main result is a representation theorem which shows that, for a large class of quantum logics, a quantum logic, Q, is isomorphic to the lattice of projective faces in a suitable convex set K. As an application we extend our earlier results [4], which, subject to countability conditions, gave a geometric characterization of those quantum logics which are isomorphic to the projection lattice of a von Neumann algebra or a JBW-algebra.

Introduction

Quantum logics, alias complete orthomodular lattices, arise in mathematical models of quantum mechanics as the lattice of "questions." The completely additive probability measures on a quantum logic correspond to the physical states of a quantum mechanical system. When M is a von Neumann algebra or a JBW-algebra then $P(M)$, the lattice of projections in M, is a quantum logic. For mathematical convenience, rather than any compelling physical reason, it is frequently assumed that the quantum logic of a mechanical system is $P(M)$. So it may be of some physical as well as mathematical interest to determine when a given quantum logic is isomorphic to a lattice of projections, $P(M)$.

Atomic lattices have been very thoroughly investigated by a number of authors and, in particular, key results were obtained by Piron and Wilbur. For a detailed discussion and a full list of references, the reader is referred to [11]. Using projective geometry, Wilbur characterized $P(L(H))$ by lattice conditions, [11]. Unfortunately, these methods are not applicable to non-atomic lattices. One way of coping with this difficulty is to focus attention on the geometry of the convex set of completely additive probability measures (physical states). Using this approach, we characterised the lattices $P(M)$ where M is a von Neumann algebra or a JBW-algebra, up to isomorphism, subject to certain countability conditions, see [4]. We made use of a deep theorem of Iochum and Shultz, [9], on the geometry of the normal state spaces of von Neumann algebras and JBW-algebras, together with the Gleason–Christensen–Yeadon theorem [5, 12] and its extension to JBW-algebras [3].

Our results here show that a very large class of quantum logics may be identified with lattices of "projections" arising as natural geometric objects in certain convex