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Non-linear σ-Models on Compact Riemann Surfaces

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Abstract. The classical $O(3)$ non-linear σ-model is generalised to a theory of fields defined on a compact Riemann surface M with values in a compact Kähler manifold V. The dimension of the space of self-dual fields from M to the complex projective space $\mathbb{P}^N$ is calculated and the classifying space for the inequivalent quantisations of the theory is also calculated.

1. Introduction

The main reason for studying the classical $O(3)$ non-linear σ-model in two dimensions is its similarities with pure Yang–Mills theory in four dimensions. The $O(3)$ model [1] is a theory of a smooth three component real field $\underline{\phi} = (\phi^a)$ $(a = 1, 2, 3)$ defined on $\mathbb{R}^2$, i.e. $\underline{\phi}:\mathbb{R}^2 \to \mathbb{R}^3$ is a smooth map. The action of the theory is

$$S[\underline{\phi}] = \tfrac{1}{2} \int_{\mathbb{R}^2} \partial_\mu \underline{\phi} \cdot \partial^\mu \underline{\phi} d^2x = \tfrac{1}{2} \int_{\mathbb{R}^2} \delta^{\mu\nu} \partial_\mu \phi^a \partial_\nu \phi^a d^2x, \tag{1.1}$$

where $\delta^{\mu\nu}$ is the Euclidean metric on $\mathbb{R}^2$. The field $\underline{\phi}$ is subject to the constraint

$$\underline{\phi}^2 \equiv \phi^a \phi^a = 1. \tag{1.2}$$

The action (1.1) is invariant under a conformal change in the metric

$$g_{\mu\nu} = \Omega^2 \delta_{\mu\nu} \tag{1.3}$$

for Ω a smooth real-valued function on $\mathbb{R}^2$. Taking

$$\Omega = 2/(1 + x^2) \tag{1.4}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, and assuming that the field $\underline{\phi}$ obeys the boundary condition

$$\underline{\phi}(x) \to \phi_\infty \quad \text{as} \quad |x| \to \infty, \tag{1.5}$$

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