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Some Considerations on the Nonlinear Stability of Stationary Planar Euler Flows*

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Abstract. We give sufficient conditions for the nonlinear stability of possibly nonsmooth stationary solutions of the two-dimensional Euler equation in symmetric bounded domains. We use, as Lyapunov functions, first integrals due to the symmetry of the problem. Moreover, we investigate the stability of smooth solutions under perturbations of the boundary. The last result is based on a generalization of the well known Arnold approach.

1.

Some years ago Arnold [1] proposed an approach to investigate the nonlinear stability of stationary Euler flows. According to the theory of finite dimensional Hamiltonian systems, the basic idea was to look for conditions ensuring the vanishing of the first variation of the energy functional and the positivity of its second variation. We briefly review the argument. For a more complete analysis we address the reader to [2], where other infinite-dimensional situations are also discussed.

Consider an incompressible ideal fluid contained in a domain D bounded by two smooth curves $\mathscr{C}_1$ and $\mathscr{C}_0$, which are the internal and external boundary respectively. Then the following functional

$$\hat{H}=\frac{1}{2}\int_D u^2 dx dy+\int_D \Phi(\omega) dx dy+\sum_{i=0}^{1} a_i \oint_{\mathscr{C}_i} u\cdot d\ell \tag{1.1}$$

(u is the velocity field, $\omega=\operatorname{curl} u=\partial_x u^{(2)}-\partial_y u^{(1)}$, Φ a real valued function, a_i real numbers) is a constant of motion, each of the three terms appearing in the right-hand side of (1.1) being first integrals.

The condition $\delta\hat{H}(\bar{u})=0$, for $\bar{u}$ stationary solution, yields (see [2] for details)

$$a_i=-\Phi'(\bar{\omega}|_{\mathscr{C}_i}) \qquad (\bar{\omega}=\operatorname{curl}\bar{u}), \tag{1.2}$$

$$\bar{u}=\nabla^{\perp}\Phi'(\bar{\omega}) \qquad \nabla^{\perp}=(\partial_y,-\partial_x). \tag{1.3}$$

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