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Instability of Nonlinear Bound States

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Abstract. We establish a sharp instability theorem for the bound states of lowest energy of the nonlinear Klein–Gordon equation, $u_{tt} - \Delta u + f(u) = 0$, and the nonlinear Schrödinger equation, $-iu_t - \Delta u + f(u) = 0$.

Introduction

In this paper we prove a sharp instability theorem for the bound states of lowest energy of the nonlinear Klein–Gordon equation

(NLKG) $$u_{tt} - \Delta u + f(u) = 0$$

and the nonlinear Schrödinger equation

(NLS) $$-iu_t - \Delta u + f(u) = 0.$$

By a bound state we mean a solution of the form

$$u(x,t) = e^{i\omega t}\phi_\omega(x)$$

with ω a real parameter and $\phi_\omega(x)$ suitably vanishing as $|x| \to \infty$. The nonlinearity f is very general: it satisfies conditions which are sufficient and are almost necessary for the existence of such bound states. Each of these systems is distinguished by a pair of invariants, energy E and charge Q. Energy comes from the time invariance and charge from the gauge invariance. It is the gauge invariance which makes the problem interesting. It means that $\phi_\omega(x)\exp i(\omega t + \theta)$ is a bound state for any constant θ. Our original proof of instability for NLKG also made use of the broken dilation invariance. However, this method gave a sharp result only in the case of a pure power nonlinearity $f(u) = u - |u|^{p-1}\, u$, $p > 1$ (cf. Sect. 4). The general case requires a more abstract method.

In general the states are stable for some ω and unstable for other ω. Stability

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