Commun. Math. Phys. 97, 149–159 (1985)

Communications in Mathematical Physics

Indecomposable Representations with Invariant Inner Product

A Theory of the Gupta-Bleuler Triplet

Huzihiro Araki

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

Abstract. Consequences of the existence of an invariant (necessarily indefinite) non-degenerate inner product for an indecomposable representation π of a group G on a space $\mathfrak{H}$ are studied. If π has an irreducible subrepresentation π_1 on a subspace $\mathfrak{H}_1$, it is shown that there exists an invariant subspace $\mathfrak{H}_2$ of $\mathfrak{H}$ containing $\mathfrak{H}_1$ and satisfying the following conditions: (1) the representation $\pi_1^{\#}=\pi \bmod \mathfrak{H}_2$ on $\mathfrak{H} \bmod \mathfrak{H}_2$ is conjugate to the representation $(\pi_1, \mathfrak{H}_1)$, (2) $\mathfrak{H}_1$ is a null space for the inner product, and (3) the induced inner product on $\mathfrak{H}_2 \bmod \mathfrak{H}_1$ is non-degenerate and invariant for the representation

$$\pi_2=(\pi|_{\mathfrak{H}_2}) \bmod \mathfrak{H}_1\,,$$

a special example being the Gupta-Bleuler triplet for the one-particle space of the free classical electromagnetic field with $\mathfrak{H}_1=$space of longitudinal photons and $\mathfrak{H}_2=$the space defined by the subsidiary condition.

1. Introduction

In the study of massless particles, one meets [1–6, 9–11, 15, 17–19] indecomposable representations π of a Lie group G on a space $\mathfrak{H}$ (with an invariant indefinite inner product), which take, for example, the following form:

$$\left.\begin{array}{c}\pi_n \to \pi_{n-1} \to \dots \to \pi_1 \\ \mathfrak{H}=\mathfrak{H}_n \supset \mathfrak{H}_{n-1} \supset \dots \supset \mathfrak{H}_1\end{array}\right\}. \tag{1.1}$$

Here $\mathfrak{H}_j$ is a $\pi(G)$-invariant subspace of $\mathfrak{H}_{j+1}$ without any $\pi(G)$-invariant complement, namely, there are no subspaces $\mathfrak{H}'_j$ such that $\mathfrak{H}_j+\mathfrak{H}'_j=\mathfrak{H}_{j+1}$, $\mathfrak{H}_j \cap \mathfrak{H}'_j=\mathbf{0}$, $\pi(G)\mathfrak{H}'_j \subset \mathfrak{H}'_j$. The representation π_j of G on $\mathfrak{H}_j/\mathfrak{H}_{j-1}$ is obtained by first restricting π to the subspace $\mathfrak{H}_j$ of $\mathfrak{H}$ and then considering it modulo $\mathfrak{H}_{j-1}: \pi_j(g)[\xi]=[\pi(g)\xi]$ for $\xi \in \mathfrak{H}_j$, where $[\xi]=\xi+\mathfrak{H}_{j-1}$ is a vector in $\mathfrak{H}_j/\mathfrak{H}_{j-1}$. (We note that the construction of representations on a quotient space, especially with respect to a null space, is now a popular game [14].)