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Percival Variational Principle for Invariant Measures and Commensurate-Incommensurate Phase Transitions in One-Dimensional Chains

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Abstract. We prove for a one-dimensional system of classical particles with potential energy,

$$U_{\alpha,\gamma} = \sum_n [\alpha V(x_n) + F(x_{n+1} - x_n - \gamma)],$$

the existence of such a smooth function $\gamma(\alpha)$, $0 \leqq \alpha \leqq \alpha_0(\omega)$ that the system with potential energy $U_{\alpha,\gamma(\alpha)}$ has the equilibrium state at the temperature $T=0$. This is the incommensurate phase with the ratio of periods equal to the prescribed irrational number ω, badly approximated by rational ones. A simple geometric condition for the invariant curve of the corresponding dynamical system is established under which it is the support of the invariant measure minimizing Percival's energy functional.

1. Introduction

The main result of the paper contains the solution of the problem stated in [1, 2] and concerns commensurate-incommensurate phase transitions in one-dimensional chains. The potential energy for the system has the form

$$U_{\alpha,\gamma} = \sum_n [\alpha V(x_n) + F(x_{n+1} - x_n - \gamma)]. \tag{1.1}$$

Here x_n are the coordinates of particles, $V(x)$ is a periodic function with period 1 having nondegenerate minima at $x=n$ and maxima at $x=n+\frac{1}{2}$, $n \in \mathbb{Z}$, F is the potential energy of the inner interaction between nearest neighbours, and α and β are parameters. We assume also that F is strictly convex, $F'' \geqq \text{const} > 0$, $F(0) = F'(0) = 0$.

As it was shown in [1] the phase diagram of the model for the temperature $T=0$ is described in terms of invariant measures of mappings of the two-dimensional cylinder $C = S^1 \times \mathbb{R}$. The transformation for (1.1) is defined as follows: $f(x, y) = (x', y')$, where

$$y = -\alpha V'(x) + F'(x' - x - \gamma),$$
$$y' = F'(x' - x - \gamma). \tag{1.2}$$