Commun. Math. Phys. 94, 115–132 (1984)

Communications in Mathematical Physics

Oscillatory Critical Amplitudes in Hierarchical Models

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Abstract. We study the oscillatory critical amplitudes of the q-states Potts model on a diamond hierarchical lattice. We consider an example of the generic case (finite critical index), as well as the degenerate case (essential singularity). In both cases, we compare the magnitude of the oscillations with geometrical characteristics of the Julia set of zeroes of the partition function.

1. Introduction

Much interest has been recently devoted to statistical mechanical models on hierarchical lattices [1–10]. On such lattices, the models can be solved by writing an exact renormalization group transformation. In particular one can compute exactly the free energy and all its derivatives. One can also know the whole set of singularities of this free energy in the complex temperature plane. It has been shown for many examples [7–10] that these singularities are located on the Julia set [10–16] associated with the renormalization transformation, i.e. the smallest closed set which contains all the unstable periods and their preimages. In the simplest cases [10], where one expects a single phase transition, the Julia set contains only a few points on the real axis. On the contrary, in some frustrated cases [5–9], one finds that the Julia set has an infinite number of points on the real axis, giving rise to an infinite number of critical temperatures [7].

Recently a relationship [17] between the shape of the density of zeroes, the critical exponents and the critical amplitudes has been proposed. This was a motivation for the study of analogous relationships in the case of hierarchical models. In this case, the critical exponent (which characterizes the singular behavior of the free energy at a critical point) can be easily extracted by linearizing the renormalization transformation at the fixed point. The shape of the Julia set can also be found easily [10]. As we shall see, the critical amplitudes are more difficult to obtain. It has been observed [7–10] in many cases that the critical behavior on hierarchical lattices is modulated: instead of observing as a leading singularity a pure power law, one finds a power law multiplied by a periodic function of the logarithm of the distance to the critical point.

The purpose of the present paper is to relate these oscillatory amplitudes to the shape of the Julia set near the critical point.

We shall consider successively two cases: the generic one, where the critical point is an isolated fixed point, and the degenerate one, where the critical point is a fixed point of multiplicity two. In both cases we shall determine numerically the shape of the Julia set and the oscillatory critical amplitudes. In the degenerate case,