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Instantons in Two and Four Dimensions

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Abstract. It is shown that Yang-Mills instantons in four dimensions can naturally be identified with the instantons of a two-dimensional theory with values in the loop group.

1. Introduction

Because of the daunting difficulties involved in attempting to quantize realistic physical gauge-theories in four-dimensional space-time considerable attention has been given to certain two-dimensional (2D) models, which it is hoped share some of the important qualitative features of the four-dimensional (4D) theories. In particular pure Yang-Mills theory in 4D is compared with the CP_n-models in 2D. Both theories are conformally invariant and possess instantons, and this provides a basis for obvious analogies.

The purpose of this paper is to strengthen the analogy concerning the instantons in these two theories. Essentially we shall show (at least for G a classical group and probably for all G) that Yang-Mills instantons in 4D can be naturally identified with (i.e. have the same parameter space as) the instantons in 2D for the theory in which the complex projective n-space CP_n is replaced by the *infinite-dimensional* manifold ΩG of loops on the structure group G. Such a theory is not as bizarre as it appears because ΩG is well-known to share most of the important properties of CP_n and it arises naturally in many contexts.

A natural identification between the instantons of two different theories suggests that there might be a close relation between the two field-theories involved. Our result therefore indicates that it would be worth exploring the two-dimensional theory for ΩG-valued fields, and that this might provide a bridge between the CP_n-models in 2D and Yang-Mills theory in 4D.

It is well-known that the CP_n-instantons are given by holomorphic (or rational) maps $CP_1 \to CP_n$, where $CP_1 = R^2 \cup \infty$ is the conformal compactification of R^2. This depends on the fact that CP_n is a Kähler manifold. Now ΩG is an infinite-dimensional Kähler manifold [15, 17] and so ΩG-instantons are also given