Commun. Math. Phys. 92, 555–594 (1984)

Communications in Mathematical Physics

Small $\hbar$ Asymptotics for Quantum Partition Functions Associated to Particles in External Yang-Mills Potentials

Robert Schrader[1, 2, ⋆, †] and Michael E. Taylor[1, ⋆⋆]

1 Mathematics Department, SUNY, Stony Brook, NY 11794, USA
2 Institute for Theoretical Physics, SUNY, Stony Brook, NY 11794, USA

Abstract. To a gauge field on a principal G-bundle $P \to M$ is associated a sequence of quantum mechanical Hamiltonians, as Planck's constant $\hbar \to 0$ and a sequence of representations π_n of G is taken. This paper studies the associated quantum partition functions, trace $\exp(-tH_n)$, and produces a complete asymptotic expansion, as $\hbar \to 0$, $\hbar = 1/n$, of which the principal term, proportional to the classical partition function, is the familiar classical limit.

1. Introduction

In this paper we study the limit as $\hbar \to 0$ of the (non-relativistic) quantum partition function associated with the Hamiltonian for motion in a Yang-Mills field. More specifically, let M be a compact Riemannian manifold, and let $P \to M$ be a principal G-bundle, G a compact connected Lie group. We suppose a connection is given on P; this determines a gauge field. We can regard the connection as a $\mathfrak{g}$-valued one-form θ. We have an associated covariant derivative on any associated vector bundle $E = P \times_\pi V$, where π is a representation of G on a vector space V. With respect to a local frame, this is given by

$$\nabla_X^\pi u = X \cdot u + \pi(\theta(X))u, \tag{1.1}$$

where X is a tangent vector to M, $u \in C^\infty(M, E)$. Here $X \cdot u$ represents the action of X componentwise on u, and $\theta(X)$ is the element of $\mathfrak{g}$ defined by the connection 1-form θ. In local coordinates, on a coordinate patch $\mathcal{O} \subset M$, with $X = \partial/\partial x_j = \partial_j$ and

$$\theta = \sum A_j(x) dx_j; \qquad A_j \in C^\infty(\mathcal{O}, \mathfrak{g}), \tag{1.2}$$

⋆ Research supported in part by Deutsche Forschungsgemeinschaft and NSF grant NoPhy 81-09011A-01

† On leave of absence from Freie Universität, Berlin

⋆⋆ Research supported by NSF grant MCS 820176A01