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Pseudodifferential Operators on Supermanifolds and the Atiyah–Singer Index Theorem*

Ezra Getzler

Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

Abstract. Fermionic quantization, or Clifford algebra, is combined with pseudodifferential operators to simplify the proof of the Atiyah–Singer index theorem for the Dirac operator on a spin manifold.

Introduction

Recently, an outline of a new proof of the Atiyah–Singer index theorem has been proposed by Alvarez-Gaumé [1], extending unpublished work of Witten. He makes use of a path integral representation for the heat kernel of a Hamiltonian that involves both bosonic and fermionic degrees of freedom. In effect, a certain democracy is created between the manifold and the fermionic variables corresponding to the exterior algebra $\Lambda^* T_x^* M$ at each point $x \in M$. In this paper, this idea is pursued within the context of Hamiltonian quantum mechanics, for which the powerful calculus of pseudodifferential operators exists, permitting a rigorous treatment of Alvarez-Gaumé's ideas.

Most of the work will be to unify pseudodifferential operators with their fermionic equivalent, the sections of a Clifford algebra bundle over the manifold. Using a pseudodifferential calculus based on the papers of Bokobza-Haggiag [7] and Widom [17, 18], but incorporating a symbol calculus for the Clifford algebra as well, an explicit formula for composition of principal symbols is derived (0.7). This permits a complete calculation of the index of the Dirac operator to be performed, modeled on a proof of Weyl's theorem on a compact Riemannian manifold M.

Recall that this theorem states that

$$\operatorname{Tr} e^{-t\Delta} = \left(\frac{\pi}{t}\right)^{n/2} \operatorname{vol}(M) + O(t^{-n/2+1}). \tag{0.1}$$

Using the calculus of pseudodifferential operators, one shows that the symbol σ of the heat kernel satisfies.

$$\sigma(e^{-t\Delta}) = e^{-t|\xi|^2} + \text{small error}, \tag{0.2}$$

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