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Renormalization Group Analysis of Quasi-Periodicity in Analytic Maps*

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Abstract. Analytic maps of the form $f(z)=e^{2\pi i\Omega}z+\mathcal{O}(z^2)$ display quasiperiodicity when Ω satisfies a diophantine condition. Quasiperiodic motion is confined to a neighborhood of the origin known as a Siegel domain. The boundary of this domain obeys universal scaling relations. In this paper we investigate these scaling relations through a renormalization group analysis, and we discuss singularities and asymptotic form of the scaling function.

Introduction

The discovery of scaling laws associated with the transition from quasiperiodic to chaotic behavior is an exciting development in the theory of dynamical systems. Shenker and Kadanoff [1], following Greene's [2] methodology, found that K.A.M. tori in area preserving maps disappear by becoming nondifferentiable in a scale invariant fashion. Shenker [3] studied the analogous phenomenon in dissipative systems and found scaling behavior in maps of a circle.

These transitions lie in universality classes; many different mappings show identical scaling behavior. Thus one can use simple models to analyze the scaling behavior of complicated dynamical systems. Provided a dynamical system satisfies a few constraints, the scaling laws are independent of all other details. Renormalization group arguments, introduced into the theory of mappings by Feigenbaum [4], explain this universality. Collet and Eckmann [5] discussed this application of the renormalization group.

In a previous application of the renormalization group to quasiperiodic systems, Kadanoff [6, 7] explored universal scaling functions for K.A.M. tori. Escande and Doveil [8] implemented the renormalization group on Hamiltonians instead of maps. MacKay [9] obtained a numerical solution of the renormal-

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