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On the Initial Value Problem of the Second Painlevé Transcendent

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Abstract. The initial value problem associated with the second Painlevé Transcendent is linearized via a matrix, discontinuous, homogeneous Riemann-Hilbert (RH) problem defined on a complicated contour (six rays intersecting at the origin). This problem is mapped through a series of transformations to three different simple Riemann-Hilbert problems, each of which can be solved via a system of two Fredholm integral equations. The connection of these results with the inverse scattering transform in one and two dimensions is also pointed out.

1. Introduction

At the turn of the century Painlevé [1] and his school [2] classified all equations of the form $q_{tt} = F(q_t, q, t)$, where F is rational in q_t, algebraic in q and locally analytic in t, which have the *Painlevé property*, i.e. their solutions are free from movable critical points [3]. Distinguished amongst these fifty equations are the so-called six Painlevé transcendents, P I–P VI; any other of the fifty equations can either be integrated in terms of known functions or can be reduced to one of these six equations.

In the Soviet literature [4] interesting results regarding exact solutions and properties of the Painlevé transcendents are summarized in [5]; the main results are: i) For a certain choice of their parameters, P II–P V admit rational solutions, as well as one-parameter families of solutions expressible in terms of elementary transcendental functions (Airy, Bessel, Weber-Hermite, Whittaker, respectively). ii) P II–P V admit transformations [6–9] which map solutions of a given Painlevé equation to solutions of the same equation, but with different values of its parameters. These transformations can be used to generate recursively the solutions mentioned in i). Similar results have also been obtained for P VI [10].

In recent years further interest in the Painlevé equations has developed due to the following reasons: i) Although P I–VI were first discovered from strictly mathematical considerations, they have recently appeared in several physical